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Simultaneous Optimization of Assignments and Goal Formations for Multiple Robots

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Abstract— **This paper presents algorithms to simultaneously compute the optimal assignments and formation parameters for a team of robots from a given initial formation to a variable goal formation (where the shape of the goal formation is given, and its scale and location parameters must be optimized). We assume the n robots are identical spheres. We use the sum of squared travel distances as the objective function to be minimized, which also ensures that the trajectories are collision free. We show that this *assignment with variable goal formation problem* can be transformed to a linear sum assignment problem (LSAP) with pseudo costs that we establish are independent of the formation parameters. The transformed problem can then be solved using the Hungarian algorithm in $\mathcal{O}(n^3)$ time. Thus the assignment problem with variable goal formations using this new approach has the same $\mathcal{O}(n^3)$ time complexity as the standard assignment problem with fixed goal formations. Results from simulations on 200 and 600 robots are presented to show the algorithm is sufficiently fast for practical applications.**

I. Introduction

Teams of robots often have to move from one formation to another as they perform exploration, coverage, and surveillance tasks [1]. Such application scenarios are becoming increasingly common as the cost of robots continues to drop. This paper presents algorithms to compute the optimal assignments and formation parameters for a team of robots from a given initial formation to a *variable goal formation*; here by variable goal formation we mean the desired shape of the goal formation is given, and its scale and location parameters can be varied. We use a minimum sum of squared distances objective to ensure that the resulting trajectories are collision free. Teams of unmanned aerial vehicles (UAVs) or ground mobile robots often need to change formations in order to navigate through narrow passages in an environment with obstacles. Most work uses a single predefined goal formation [1] for the team of robots or selects from a set of predefined formations based on the route. Such approaches do not exploit the additional flexibility for the goal formation that is often possible — the formation could be scaled or its location may be changed to optimize the objective function. While efficient algorithms for variable formations with fixed assignments are presented in [2], there is only limited prior work where both the assignment and the variable formation are considered simultaneously [3]. This is precisely the gap that this paper addresses. Further, we show that this variable goal formation problem can be transformed to a linear sum assignment problem (LSAP) [4], which can then be solved using the Hungarian algorithm in $\mathcal{O}(n^3)$ time, just like the standard assignment problem with a fixed goal formation.

The algorithms presented here can also benefit an emerging novel application: the programming of large teams of mobile robots [5] or UAVs [6] to create animated light shows with LEDs mounted on the robots. Here the synchronized robot formations create visual images for entertainment. Recently Intel has demonstrated large fleets of its Shooting Star drones (numbering 300–500 drones) for such visual performances [7]. The quadrotor drones have LED lights that change color and intensity to create appealing 3D displays. The robots must be assigned goal locations and moved to them along generated collision-free trajectories. These algorithms would also benefit nanosatellite swarm formations requiring reconfiguration [8].

Another application domain these algorithms are designed to address is droplet-based lab-on-chip systems for point-of-care medical diagnostics. In these light-actuated digital microfluidic (LADM) systems, discrete droplets of chemicals are optically actuated using moving patterns of projected light to perform chemical reactions by repeatedly moving droplets to mixing formations (e.g., [9], [10], [11], [12]). By modeling the droplets as robots, we can address the automated coordination of droplets on the LADM chip, including determining goal formations that can fit within specified regions of the chip.

II. Related Work

Multi-robot assignment and trajectory planning: Multi-robot assignment and task allocation has been an area of active research; see Gerkey and Mataric [13] and Dias et al. [14] for surveys. We first review assignment of multiple robots to fixed formations. Kloder and Hutchinson [15] developed a representation for collision-free path planning of multiple unlabeled robots translating in the plane between two given formations. They represent a formation by the coefficients of a complex polynomial whose roots represent the robot configurations. Turpin, Michael, and Kumar [16] presented an algorithm for generating robot assignments and trajectories for a team of robots moving from an initial formation to a fixed goal formation. The robots are assumed to be identical with equal radii. They show that by minimizing the sum of the squares of velocities, the generated trajectories have constant velocities and the resulting assignment guarantees that there is no collision between the robots, under certain initial separation conditions; we build

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on this result. MacAlpine, Price, and Stone [17] developed a collision-avoiding assignment algorithm that minimizes the maximum robot travel distance for fixed goal formations. Morgan et al. [8] developed a distributed auction algorithm for spacecraft swarm assignment and collision-free trajectory optimization. Recently Preiss et al. [18] presented an assignment and trajectory optimization algorithm for quadrotors in the presence of obstacles and downwash constraints. It uses a spatial grid to generate discretized trajectories and subsequently refines them into continuous trajectories.

Variable formations with fixed assignments: Derenick and Spletzer [2] presented the first algorithm to find optimal parameters for variable goal formations; however it assumes the robot assignment is given. The scale, orientation, and translation are all treated as parameters. Using second-order cone programming techniques for solving the optimization problem, they show that the theoretical complexity of the algorithm is $\mathcal{O}(n^{1.5})$, and linear in practice.

Assignment for variable formations: Akella [3] presented the first algorithms for simultaneously solving the assignment problem and optimizing the formation parameters for variable goal formations. Here scale and translation are considered, although separately. The problem minimizes the completion time, and is solved as a linear bottleneck assignment problem (LBAP). Since the LBAP solution depends on the order of the costs rather than their actual values, computational geometry techniques are used to find the optimal solution.

Formation changes in the presence of obstacles: Alonso-Mora et al. [19] present algorithms to find convex regions within the free space of an environment with obstacles, and then use a centralized method for navigating a team of robots in formation. The obstacles may be static and/or dynamic. Sequential convex programming is used to find the optimal parameters for the formation. The individual robots then use a local planner to avoid any further collision and account for the dynamics.

III. The Assignment with Variable Goal Formation Problem

Let there be n identical spherical robots with equal radius R. The initial positions of the robots are given by $\mathbf{p}_i = (p_{ix}, p_{iy}, p_{iz})^\top, i = 1, \ldots, n$. Let the initial formation be represented by $\mathbf{P} = (\mathbf{p}_i^\top)$, an $n \times 3$ matrix. The desired shape is given by n positions $\mathbf{s}_j = (s_{jx}, s_{jy}, s_{jz})^\top, j = 1, \ldots, n$, defined in a local frame such that $\mathbf{s}_1 = (0, 0, 0)^\top$ and with axes parallel to the global frame. Let the desired shape formation be represented by $\mathbf{S} = (\mathbf{s}_i^\top)$, an $n \times 3$ matrix. The goal formation, which is to be computed based on the optimality criterion, consists of goal positions $\mathbf{q}_j = (q_{jx}, q_{jy}, q_{jz})^\top, j = 1, \ldots, n$. Let the goal formation be represented by $\mathbf{Q} = (\mathbf{q}_j^\top)$, an $n \times 3$ matrix. A translation vector $\mathbf{d} = (d_x, d_y, d_z)^\top$ is defined such that $\mathbf{q}_1 = \mathbf{d}$. The robots are assumed to move on straight line paths in an obstacle-free environment. The objective is to simultaneously assign the robots from the initial formation $\mathbf{P}$ to the goal formation $\mathbf{Q}$ and to find the parameters describing the goal formation. Further, trajectories need to be generated such that there are no collisions between the robots.

Selecting the cost as the sum of squared distances enables generation of trajectories that are collision free under the condition that the separation of robots in the initial formation and the goal formation is at least $2\sqrt{2}R$, as shown in [16]. Furthermore, the trajectories are such that the robots move with constant velocities, and all robots simultaneously start and reach their goal positions.

The objective of the paper is to develop an algorithm to compute the optimal assignments and formation parameters while also ensuring that the trajectories are collision free. An illustrative example is shown in Fig. 1.

The following subsections describe the goal formation parameters considered, and their corresponding cost functions.

A. Variable Scale

The goal formation positions can be written in terms of the scale parameter, $\alpha \in (0, \infty)$, as:

$$\mathbf{q}_j = \alpha \mathbf{s}_j + \mathbf{d} \tag{1}$$

The cost c_{ij}^α is given by the squared distance between the initial position $\mathbf{p}_i$ and the goal position $\mathbf{q}_j$, which in turn is a function of the scale parameter α.

$$\begin{aligned} c_{ij}^\alpha &= ||\mathbf{p}_i - \mathbf{q}_j||_2^2 = (\mathbf{p}_i - \alpha\mathbf{s}_j - \mathbf{d})^\top(\mathbf{p}_i - \alpha\mathbf{s}_j - \mathbf{d}) \\ &= (\mathbf{p}_i - \mathbf{d})^\top(\mathbf{p}_i - \mathbf{d}) - 2\alpha(\mathbf{p}_i - \mathbf{d})^\top\mathbf{s}_j + \alpha^2\mathbf{s}_j^\top\mathbf{s}_j \end{aligned} \tag{2}$$

B. Variable Translation

For variable translation $\mathbf{d}$, the formation positions are:

$$\mathbf{q}_j = \mathbf{s}_j + \mathbf{d} \tag{3}$$

The cost c_{ij}^d is given by:

$$\begin{aligned} c_{ij}^d &= ||\mathbf{p}_i - \mathbf{q}_j||_2^2 = (\mathbf{p}_i - \mathbf{s}_j - \mathbf{d})^\top(\mathbf{p}_i - \mathbf{s}_j - \mathbf{d}) \\ &= \mathbf{p}_i^\top\mathbf{p}_i + \mathbf{s}_j^\top\mathbf{s}_j - 2\mathbf{p}_i^\top\mathbf{s}_j - 2(\mathbf{p}_i - \mathbf{s}_j)^\top\mathbf{d} + \mathbf{d}^\top\mathbf{d} \end{aligned} \tag{4}$$

C. Variable Scale and Translation

For variable scale α and translation $\mathbf{d}$, the formation positions are given by:

$$\mathbf{q}_j = \alpha \mathbf{s}_j + \mathbf{d} \tag{5}$$

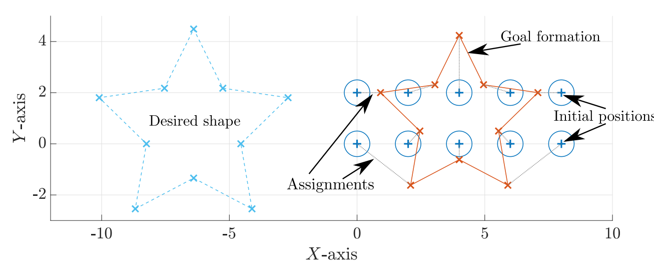

Fig. 1. Illustrative example with ten robots. The circular robots are in two parallel rows in their initial formation. The desired formation shape is a star. The objective is to compute the optimal assignment and the optimal parameters (scale, translation) of the goal formation (indicated by the star with red lines). The assignments are shown by dotted lines. For clarity, the robots are not depicted at their goal positions.

The cost $c_{ij}^{\alpha d}$ is given by:

$$\begin{aligned} c_{ij}^{\alpha d} &= ||\mathbf{p}_i - \mathbf{q}_j||_2^2 = (\mathbf{p}_i - \alpha\mathbf{s}_j - \mathbf{d})^\top(\mathbf{p}_i - \alpha\mathbf{s}_j - \mathbf{d}) \\ &= \mathbf{p}_i^\top\mathbf{p}_i + \alpha^2\mathbf{s}_j^\top\mathbf{s}_j - 2\alpha\mathbf{p}_i^\top\mathbf{s}_j + 2\alpha\mathbf{s}_j^\top\mathbf{d} \\ &\quad - 2\mathbf{p}_i^\top\mathbf{d} + \mathbf{d}^\top\mathbf{d} \end{aligned} \tag{6}$$

IV. Simultaneous Assignment and Goal Formation Parameter Optimization

The problem of simultaneously computing the optimal assignment and formation parameters for multiple robots, while minimizing the sum of squared distances, can be posed as the following optimization problem:

$$\begin{aligned} \text{Minimize} \qquad & \mathcal{C} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} = 1 \quad i = 1, \ldots, n \\ & x_{ij} = \{0, 1\} \quad i, j = 1, \ldots, n \end{aligned} \tag{7}$$

The variables c_{ij} represent the cost of assigning robot i to position j, and are functions of the formation parameters such as $\alpha, \mathbf{d}$, depending on the requirements. The binary variables x_{ij} represent the assignment of the ith robot to the jth goal position. Let the assignment matrix be $\mathbf{X} = (x_{ij})$. We first establish some preliminary results.

We now show that even though the costs c_{ij} are functions of the formation parameters, the optimization problem stated in (7) can be transformed to a *Linear Sum Assignment Problem* (LSAP). We derive a modified (pseudo) cost function for the LSAP, the solution to which results in the same optimal assignment as the original problem. The pseudo cost function is identical for goal formations with variable scale, variable translation, and combined variable scale and translation.

Lemma 1: The double summation of the form $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i x_{ij}$, where the a_i, $i = 1, \ldots, n$ are constants depending only on the index i, is a constant.

Proof: The double summation can be simplified, using $\sum_{j=1}^{n} x_{ij} = 1$ from (7), as:

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} a_i x_{ij} &= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} x_{ij} \\ &= \sum_{i=1}^{n} a_i = \text{constant.} \end{aligned} \tag{8}$$

■

Lemma 2: The double summation of the form $\sum_{i=1}^{n}\sum_{j=1}^{n} b_j x_{ij}$, where the b_j, $j = 1, \ldots, n$ are constants depending only on the index j, is a constant.

Proof: Similar to Lemma 1, the double summation can be simplified as:

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} b_j x_{ij} &= \sum_{j=1}^{n} b_j \sum_{i=1}^{n} x_{ij} \\ &= \sum_{j=1}^{n} b_j = \text{constant.} \end{aligned} \tag{9}$$

■

Lemma 3: The double summation $\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}$ is equal to n.

Proof: From Lemma 1,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{n} 1 = n \tag{10}$$

■

A. Variable Scale

The objective function is based on minimization of the sum of squared distances, and is a function of the scale parameter α and assignment $\mathbf{X}$:

$$\mathcal{C}_\alpha(\alpha, \mathbf{X}) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^{\alpha} x_{ij}$$

The coefficients c_{ij}^{α} are given by (2). The cost $\mathcal{C}_\alpha$ is:

$$\begin{aligned} \mathcal{C}_\alpha(\alpha, \mathbf{X}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \big[&(\mathbf{p}_i - \mathbf{d})^\top(\mathbf{p}_i - \mathbf{d})x_{ij} \\ &-2\alpha(\mathbf{p}_i - \mathbf{d})^\top\mathbf{s}_j x_{ij} + \alpha^2\mathbf{s}_j^\top\mathbf{s}_j x_{ij}\big] \\ = d_{pd}^2 + 2\alpha\mathbf{d}^\top\mathbf{s} + \alpha^2 d_s^2 &+ 2\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}\left(-\mathbf{p}_i^\top\mathbf{s}_j x_{ij}\right) \end{aligned} \tag{11}$$

with the following constants derived using Lemmas 1 and 2:

$$\begin{aligned} d_{pd}^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbf{p}_i - \mathbf{d})^\top(\mathbf{p}_i - \mathbf{d})x_{ij} \\ &= \sum_{i=1}^{n}(\mathbf{p}_i - \mathbf{d})^\top(\mathbf{p}_i - \mathbf{d}) \\ d_s^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{s}_j^\top\mathbf{s}_j x_{ij} = \sum_{j=1}^{n}\mathbf{s}_j^\top\mathbf{s}_j \\ \mathbf{s} &= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{s}_j x_{ij} = \sum_{j=1}^{n}\mathbf{s}_j \end{aligned}$$

The plot in Fig. 2 shows the cost curves for all permutations of the example of Fig. 3. The optimal assignment corresponds to the lowest cost curve. Note that the cost curves do not intersect for positive values of α.

Lemma 4: If two different assignments $\mathbf{X}^1 = (x_{ij}^1)$ and $\mathbf{X}^2 = (x_{ij}^2)$ have the same cost value at some value of $\alpha > 0$, they have the same costs at all values of α.

Proof: Consider the two different assignments, $\mathbf{X}^1 = (x_{ij}^1)$ and $\mathbf{X}^2 = (x_{ij}^2)$ with the corresponding cost functions

$\mathcal{C}_\alpha(\alpha, \mathbf{X}^1)$ and $\mathcal{C}_\alpha(\alpha, \mathbf{X}^2)$. The two cost functions intersect at some value of α. Leaving aside the trivial case when $\alpha = 0$, simplifying the equation $\mathcal{C}_\alpha(\alpha, \mathbf{X}^1) = \mathcal{C}_\alpha(\alpha, \mathbf{X}^2)$ shows that these two curves intersect when:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(-\mathbf{p}_i^\top \mathbf{s}_j x_{ij}^1) = \sum_{i=1}^{n}\sum_{j=1}^{n}(-\mathbf{p}_i^\top \mathbf{s}_j x_{ij}^2) \tag{12}$$

Since (12) is independent of α, the equation $\mathcal{C}_\alpha(\alpha, \mathbf{X}^1) = \mathcal{C}_\alpha(\alpha, \mathbf{X}^2)$ will be true at any value of α. Hence the two cost curves are identical. (If $\mathbf{X}^1$ is the optimal assignment, then $\mathbf{X}^2$ is also optimal with the same cost.) ■

The above lemma establishes that an optimal cost curve does not intersect with a non-optimal one since intersection of an optimal cost curve with another cost curve implies coincidence of the two curves. Coincidence of cost curves can potentially lead to multiple optimal solutions. One example scenario with multiple optimal solutions is when all $\mathbf{p}_i$ are perpendicular to all $\mathbf{s}_j$ (e.g., $\mathbf{p}_i = (p_{ix}, 0, 0)^\top$ and $\mathbf{s}_j = (0, s_{jy}, 0)^\top$). Here the cost curves are equal for all the assignments; any assignment would be optimal.

Corollary 5: The optimal assignment at a positive value of α is the optimal assignment at any positive value of α.

Proof: Since an optimal cost curve does not intersect with a non-optimal curve for $\alpha > 0$, it is optimal over the entire range of its formation parameter α. It is therefore sufficient to compute the optimal assignment at any positive value of α. ■

Further, a new assignment problem can be formulated with pseudo costs obtained solely from the assignment-dependent component of (11). The pseudo cost function is given by:

$$\mathcal{K}(\mathbf{X}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\kappa_{ij}x_{ij} \quad \text{where } \kappa_{ij} = -\mathbf{p}_i^\top \mathbf{s}_j \tag{13}$$

The costs κ_{ij} are constants and hence, the cost function $\mathcal{K}(\mathbf{X})$, along with the constraints given in (7), forms an LSAP. Let the optimal assignment obtained, using the Hungarian algorithm, be $\mathbf{X}^*$ with optimal pseudo cost $\mathcal{K}^*$. Once the optimal assignment is obtained, the values of x_{ij} can be substituted in the original cost function.

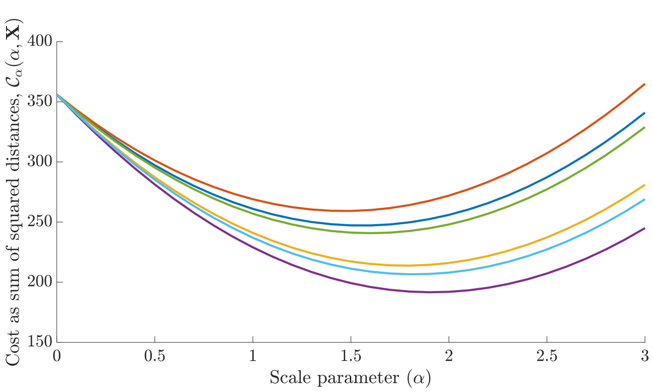

Fig. 2. Plot of cost curves $\mathcal{C}_\alpha(\alpha, \mathbf{X})$ for all the permutations of assignment $\mathbf{X} = (x_{ij})$ for the three-robot example of Fig. 3.

PSEUDO-COST-ASSIGNMENT($\mathbf{P}, \mathbf{S}, n$)

```
// P = (p_i^T) is n × 3 matrix for initial positions
// S = (s_i^T) is n × 3 matrix for desired shape positions
for i = 1 to n
    for j = 1 to n
        κ_ij = −p_i^T s_j
(X*, K*) = HUNGARIAN-LSAP(κ_ij)
return (X*, K*)
```

Convexity of the cost function: At the optimal assignment $\mathbf{X}^*$, the cost is given by:

$$\mathcal{C}_\alpha(\alpha, \mathbf{X}^*) = d_{pd}^2 + 2\alpha\mathbf{d}^\top\mathbf{s} + 2\alpha\mathcal{K}^* + \alpha^2 d_s^2 \tag{14}$$

The function is a quadratic with a positive leading coefficient. Hence, it is convex. The globally optimal scale parameter, α^*, is obtained by evaluating the value at which the derivative of $\mathcal{C}_\alpha(\alpha, \mathbf{X}^*)$ vanishes.

$$\alpha^* = -\frac{\mathcal{K}^* + \mathbf{d}^\top\mathbf{s}}{d_s^2} \tag{15}$$

The globally optimal cost is then given by $\mathcal{C}_\alpha(\alpha^*, \mathbf{X}^*)$.

VARIABLE-SCALE-FORMATION($\mathbf{P}, \mathbf{S}, \mathbf{d}, n$)

```
(X*, K*) = PSEUDO-COST-ASSIGNMENT(P, S, n)
Compute α* from (15)
return ( α*, X*)
```

Computational complexity: The optimal assignment $\mathbf{X}^*$ and pseudo cost $\mathcal{K}^*$ can be obtained in $\mathcal{O}(n^3)$ using the Hungarian algorithm. The optimal scale α^* can be computed in $\mathcal{O}(nw)$ time where w is the dimensionality of the workspace. Therefore the computational complexity of solving the variable scale formation problem is $\mathcal{O}(n^3)$.

An example assignment with variable scale formation problem for $n = 3$ robots and its optimal assignment are shown in Fig. 3.

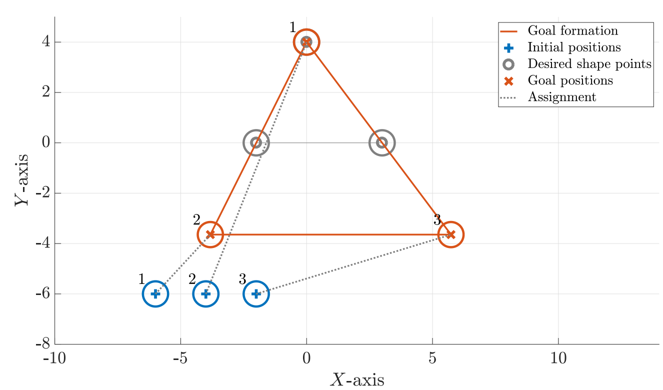

Fig. 3. An example assignment with variable scale formation problem with three robots. The radius R of the robots is 0.5 units. The initial formation of the robots is $\mathbf{P} = ((-6, -6), (-4, -6), (-2, -6))$. The desired shape is $\mathbf{S} = ((0, 0), (-2, -4), (3, -4))$. The translation parameter $\mathbf{d} = (0, 4)^\top$ is given. The optimal value of α is 1.9111, which corresponds to the lowest cost curve in Fig. 2. The goal formation positions are $q_j = \alpha\mathbf{s}_j + \mathbf{d}$.

B. Variable Translation

The objective is a function of the translation parameter $\mathbf{d}$ and the assignment $\mathbf{X}$:

$$\mathcal{C}_d(\mathbf{d},\mathbf{X}) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^{d} x_{ij}$$

The coefficients c_{ij}^{d} are given by (4). The cost $\mathcal{C}_d$ is:

$$\begin{aligned}\mathcal{C}_d(\mathbf{d},\mathbf{X}) = &\sum_{i=1}^{n}\sum_{j=1}^{n}\big[\mathbf{p}_i^\top\mathbf{p}_i x_{ij} + \mathbf{s}_j^\top\mathbf{s}_j x_{ij} - 2\mathbf{p}_i^\top\mathbf{s}_j x_{ij} \\ &\qquad -2(\mathbf{p}_i-\mathbf{s}_j)^\top\mathbf{d}x_{ij} + \mathbf{d}^\top\mathbf{d}x_{ij}\big] \\ = &\; d_p^2 + d_s^2 + n\mathbf{d}^\top\mathbf{d} - 2(\mathbf{p}-\mathbf{s})^\top\mathbf{d} \\ &+ 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(-\mathbf{p}_i^\top\mathbf{s}_j x_{ij}\right)\end{aligned} \tag{16}$$

where the following constants are derived using Lemmas 1, 2, and 3:

$$d_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{p}_i^\top\mathbf{p}_i x_{ij} = \sum_{i=1}^{n}\mathbf{p}_i^\top\mathbf{p}_i$$
$$\mathbf{p} = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{p}_i x_{ij} = \sum_{i=1}^{n}\mathbf{p}_i$$

Using (16) and reasoning similar to Lemma 4, we can show that the cost surfaces for two different permutations of the assignment will not intersect for any value of $\mathbf{d}$, unless $\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{p}_i^\top\mathbf{s}_j x_{ij}$ is equal for the two assignments; in this case the cost surfaces coincide for the two permutations at all values of $\mathbf{d}$. Using arguments similar to Corollary 5, it is sufficient to solve for the optimal assignment at any feasible value of $\mathbf{d}$. The optimal assignment corresponds to the lowest cost $C_d(\mathbf{d},\mathbf{X})$ surface.

The pseudo cost function for this LSAP is the same as that for the variable scale and is given in (13).

Convexity of the cost function: At the optimal assignment $\mathbf{X}^*$, the cost is given by:

$$\mathcal{C}_d(\mathbf{d},\mathbf{X}^*) = d_p^2 + d_s^2 + 2\mathcal{K}^* - 2(\mathbf{p}-\mathbf{s})^\top\mathbf{d} + n\mathbf{d}^\top\mathbf{d} \tag{17}$$

The Hessian of $C_d(\mathbf{d},\mathbf{X}^*)$ is a symmetric matrix with all positive and identical eigenvalues, and therefore is positive definite. The cost function is hence convex. The globally optimal translation parameter $\mathbf{d}^*$ can now be obtained by evaluating the value at which the derivative of $\mathcal{C}_d(\mathbf{d},\mathbf{X}^*)$ vanishes.

$$\mathbf{d}^* = \frac{(\mathbf{p}-\mathbf{s})}{n} \tag{18}$$

The optimal cost is given by $\mathcal{C}_d(\mathbf{d}^*,\mathbf{X}^*)$. Similar to the variable scale formation problem, the computational complexity of solving the variable translation formation problem is also $\mathcal{O}(n^3)$. An example of the optimal assignment for variable translation formation for $n = 4$ robots is shown in Fig. 4.

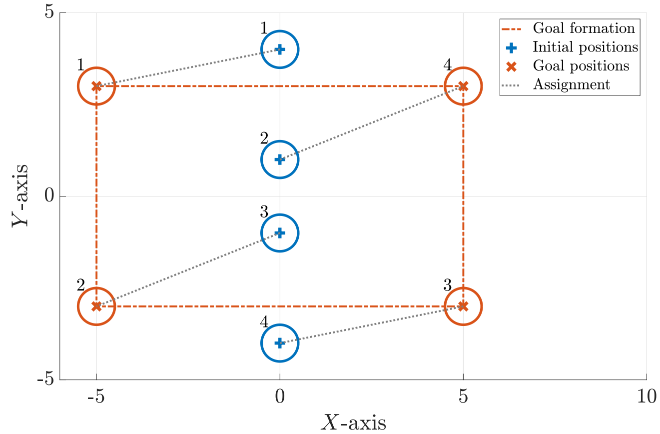

Fig. 4. An example assignment with variable translation formation problem with four robots. The radius R of the robots is 0.5 units. The desired shape is a rectangle, specified by $\mathbf{S} = ((0,0),(0,-6),(10,-6),(10,0))$. The initial formation is $\mathbf{P} = ((0,4),(0,1),(0,-1),(0,4))$. The optimal translation parameter is $\mathbf{d} = (-5,3)^\top$. The goal formation positions are $\mathbf{q}_j = \mathbf{s}_j + \mathbf{d}$.

VARIABLE-TRANSLATION-FORMATION$(\mathbf{P},\mathbf{S},n)$
$(\mathbf{X}^*,\mathcal{K}^*) =$ PSEUDO-COST-ASSIGNMENT$(\mathbf{P},\mathbf{S},n)$
Compute $\mathbf{d}^*$ from (18)
return $(\mathbf{X}^*,\ \mathbf{d}^*)$

C. Variable Scale and Translation

The objective function is a function of the scale parameter α, the translation parameter $\mathbf{d}$, and the assignment $\mathbf{X}$:

$$\mathcal{C}_{\alpha d}(\alpha,\mathbf{d},\mathbf{X}) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^{\alpha d} x_{ij}$$

The coefficients $c_{ij}^{\alpha d}$ are given by (6). Therefore

$$\begin{aligned}\mathcal{C}_{\alpha d}(\alpha,\mathbf{d},\mathbf{X}) = &\sum_{i=1}^{n}\sum_{j=1}^{n}\big(\mathbf{p}_i^\top\mathbf{p}_i x_{ij} + \alpha^2\mathbf{s}_j^\top\mathbf{s}_j x_{ij} - 2\alpha\mathbf{p}_i^\top\mathbf{s}_j x_{ij} \\ &\qquad +2\alpha\mathbf{s}_j^\top\mathbf{d}x_{ij} - 2\mathbf{p}_i^\top\mathbf{d}x_{ij} + \mathbf{d}^\top\mathbf{d}x_{ij}\big) \\ = &\; d_p^2 + \alpha^2 d_s^2 + n\mathbf{d}^\top\mathbf{d} + 2\alpha\mathbf{s}^\top\mathbf{d} \\ &- 2\mathbf{p}^\top\mathbf{d} + 2\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}\left(-\mathbf{p}_i^\top\mathbf{s}_j x_{ij}\right)\end{aligned} \tag{19}$$

We can show, using (19) and reasoning similar to Lemma 4, that the 4D cost surfaces for two different permutations of the assignment will not intersect for any positive value of α or any value of $\mathbf{d}$, unless $\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{p}_i^\top\mathbf{s}_j x_{ij}$ is equal for the two assignments; in this case the cost surfaces coincide for the two permutations at all values of α and $\mathbf{d}$. The optimal assignment for the variable scale and translation formation can therefore be solved by evaluating at any feasible $(\alpha,\mathbf{d})$. The pseudo cost function for this LSAP is same as that for variable scale and is given in (13).

Convexity of the cost function: At the optimal assignment $\mathbf{X}^*$, the cost is:

$$\mathcal{C}_{\alpha d}(\alpha, \mathbf{d}, \mathbf{X}^*) = d_p^2 + \alpha^2 d_s^2 + 2\alpha\mathcal{K}^* - 2\mathbf{p}^\top\mathbf{d} + 2\alpha\mathbf{s}^\top\mathbf{d} + n\mathbf{d}^\top\mathbf{d} \tag{20}$$

The Hessian of the cost function is symmetric with eigenvalues $\{2n, 2n, d_s^2 + n \pm \sqrt{(d_s^2+n)^2 - 4s_d}\}$, where

$$s_d = \sum_{i=1}^{n}\sum_{j=i+1}^{n} ||\mathbf{s}_i - \mathbf{s}_j||^2$$

All of the eigenvalues are nonnegative, implying a positive semidefinite Hessian matrix and so, a convex cost function.

The globally optimal scale and translation parameters, α^* and $\mathbf{d}^*$, can now be obtained by evaluating the value at which the gradient of $\mathcal{C}_{\alpha d}(\alpha, \mathbf{d}, \mathbf{X}^*)$ vanishes.

$$\begin{aligned} \alpha^* &= \frac{\mathbf{p}^\top\mathbf{s} + n\mathcal{K}^*}{\mathbf{s}^\top\mathbf{s} - nd_s^2} \\ \mathbf{d}^* &= \frac{(\mathbf{p} - \alpha^*\mathbf{s})}{n} \end{aligned} \tag{21}$$

The optimal cost is given by $\mathcal{C}_{\alpha d}(\alpha^*, \mathbf{d}^*, \mathbf{X}^*)$. Similar to the variable scale formation problem, the computational complexity of solving the combined variable scale and translation formation problem is also $\mathcal{O}(n^3)$.

VARIABLE-SCALE-TRANSLATION-FORMATION$(\mathbf{P}, \mathbf{S}, n)$
1 $(\mathbf{X}^*, \mathcal{K}^*) =$ PSEUDO-COST-ASSIGNMENT$(\mathbf{P}, \mathbf{S}, n)$
2 Compute α^* and $\mathbf{d}^*$ from (21)
3 return (α^*, $\mathbf{d}^*$, $\mathbf{X}^*$)

D. Invariance of the Optimal Assignment

Theorem 6: The optimal assignment for a given initial formation and a desired shape is invariant, and independent of the goal formation's scale and/or translation parameters.

Proof: The VARIABLE-SCALE-FORMATION, VARIABLE-TRANSLATION-FORMATION, and VARIABLE-SCALE-TRANSLATION-FORMATION algorithms all use the same PSEUDO-COST-ASSIGNMENT algorithm to compute the optimal assignment. The PSEUDO-COST-ASSIGNMENT algorithm depends only on the initial formation $\mathbf{P}$ and the desired shape $\mathbf{S}$. ■

The optimal formation parameters are computed by optimizing the appropriate cost function given in (14), (17), or (20). This also implies that for a given initial formation and desired shape, we need to compute the optimal assignment just once initially. Then given the feasible α, $\mathbf{d}$ ranges, we can compute the optimal formation rapidly in time linear in the number of robots.

V. TRAJECTORY GENERATION AND COLLISION AVOIDANCE

The robots move with constant velocity straight-line trajectories such that they start simultaneously and reach their respective goal positions simultaneously at some final time t_f. Let the maximum allowable speed for the robots be v. The final time is then given as:

$$t_f = \max_{i=1,2,\ldots,n} \frac{||\mathbf{p}_i - \mathbf{q}_{\phi(i)}||_2}{v}.$$

where $\phi(i)$ denotes the index of the goal position to which robot i is assigned, i.e., j such that $x_{ij} = 1$. The constant velocity trajectories $\mathbf{x}_i(t)$ are then given as:

$$\mathbf{x}_i(t) = \mathbf{p}_i + \left(\frac{\mathbf{q}_{\phi(i)} - \mathbf{p}_i}{t_f}\right)t, \quad t \in [0, t_f].$$

These trajectories, for an assignment that minimizes the sum of squared distances, are collision-free under the following separation conditions, defined in [16]:

$$||\mathbf{p}_i - \mathbf{p}_j||_2 > 2\sqrt{2}R \tag{22}$$
$$||\mathbf{q}_i - \mathbf{q}_j||_2 > 2\sqrt{2}R, \quad i, j = 1, 2, \ldots, n, i \neq j. \tag{23}$$

Since the initial positions of the robots are given, the user has to ensure that the distance between them is greater than $2\sqrt{2}R$, as specified in (22). For the variable translation formation, the condition in (23) becomes $||\mathbf{s}_i - \mathbf{s}_j||_2 > 2\sqrt{2}R$. Thus, the user needs to ensure that the distance between the shape positions meets the requirement.

For the case when the scale is variable, the separation condition for the goal positions, assuming $\mathbf{s}_i \neq \mathbf{s}_j$, can be written as:

$$\begin{aligned} &\alpha||\mathbf{s}_i - \mathbf{s}_j||_2 > 2\sqrt{2}R \\ \text{or, } &\alpha > \alpha^{\min} \end{aligned}$$
$$\text{where } \alpha^{\min} = \frac{2\sqrt{2}R}{\min ||\mathbf{s}_i - \mathbf{s}_j||_2} \quad \forall i, j = 1, \ldots, n, i \neq j.$$

In general, practical applications restrict the permissible values of the parameters. For example, a limit has to be placed on the maximum value of scale; limits for the translation parameters also need to be specified so that the goal formation does not lie outside the workspace. The following constraints can then be specified:

$$\mathbf{d} \in [\mathbf{d}_1, \mathbf{d}_2], \quad \alpha \in [\max(\alpha^{\min}, \alpha_1), \alpha_2]$$

where $\mathbf{d}_1, \mathbf{d}_2, \alpha_1, \alpha_2$ are limits on the parameters specified according to the application. As the objective function is convex and quadratic and the parameter constraints are linear, the minimization of the cost function (20) can be solved as a convex quadratic program (QP). Further, the KKT conditions (see e.g., [20]) provide the necessary and sufficient conditions to find the globally optimal solution.

VI. EXAMPLES

Our first example demonstrates the algorithm's ability to perform assignment with variable goal formations for a large number of robots, motivated by entertainment applications using UAVs [6], [7]. A formation of 600 identical robots of radius 0.25 units was initially set up as a rectangular grid (Fig. 5). The first desired shape consisted of the letters `UNCC`, where each letter has 150 constituent robots. Then the next desired shape was changed to `ICRA`. Computation of the optimal solution for the first formation change took about 45 seconds while the second took about 80 seconds. These computations were on a standard laptop (Intel i7-7700HQ, 2.80GHz CPU with 16GB RAM) using `MATLAB` with no performance optimization. The variable formation

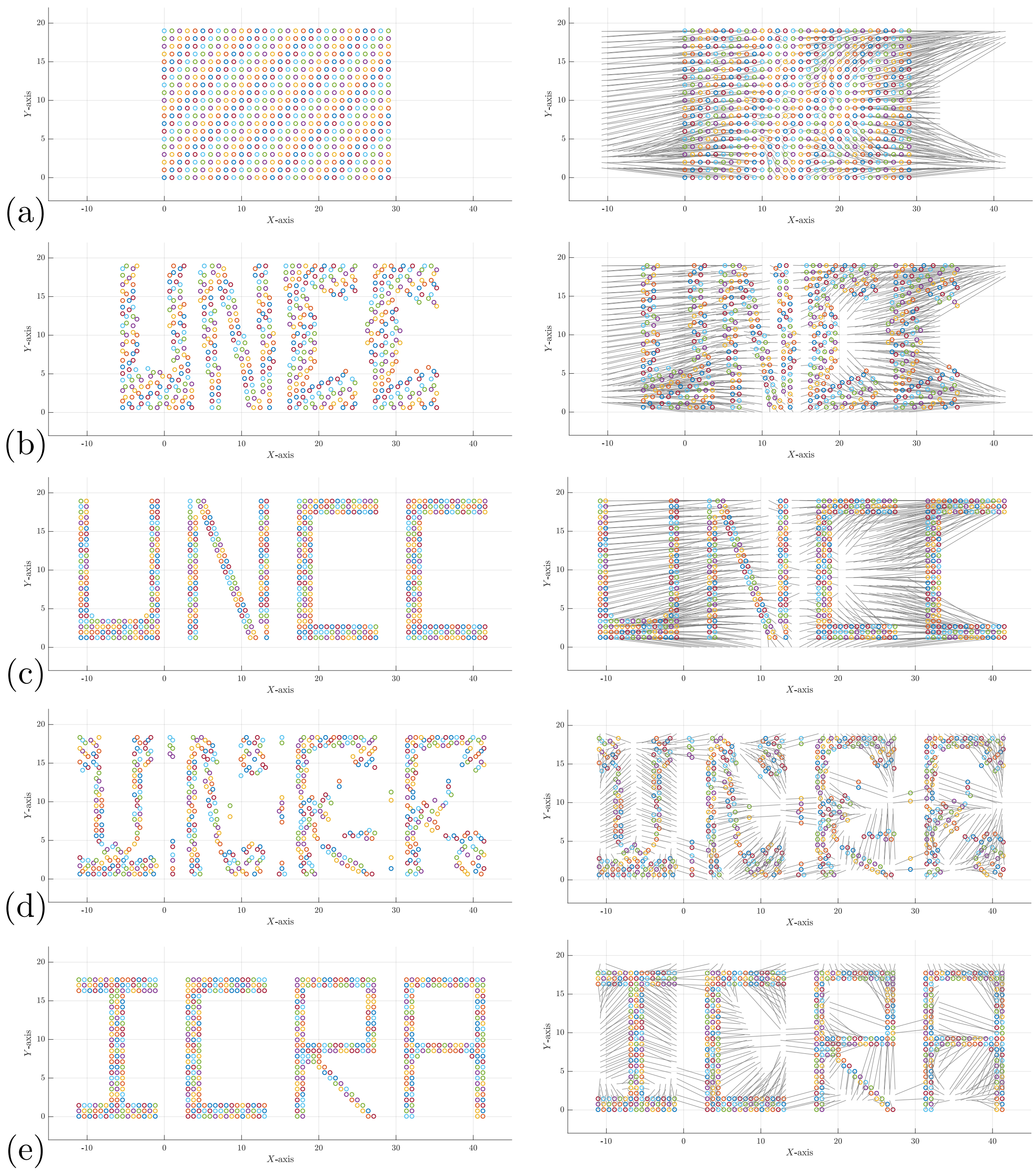

Fig. 5. Simulation of 600 robots moving from an initial formation to two successive goal formations for an assignment problem with variable scale and translation. The left column shows the formations, while the right column additionally shows the robot paths to the next formation for the computed assignments. (a) Initial formation of the robots in a rectangular grid. (b) Intermediate snapshot of the robots moving towards the `UNCC` goal formation. (c) The robots at the `UNCC` goal formation. (d) Intermediate snapshot of the robots moving towards the second goal formation of `ICRA`. (e) The robots at the `ICRA` goal formation.

parameters were scale α and translation $\mathbf{d} = (d_x, d_y)^\top$. The asymptotic complexity of the algorithm does not depend on the dimension of the parameter space, and is dominated by the complexity of the Hungarian algorithm used to solve the LSAP with the pseudo costs.

In environments with obstacles and narrow passages, robots may need to change their formations to efficiently pass through, as shown in [19]. Depending on the shape and size of the narrow passages, the desired shape is selected and a valid range of formation parameters is determined. Fig. 6 illustrates such a scenario, where 200 robots initially in cylindrical formation change to a spherical formation to pass through an opening in the wall. The feasible range of scales is decided by the aperture of the opening and the range of the translation parameter is determined by feasible locations for the goal formation. It took around 2 seconds to solve the problem.

The supplemental video contains animated simulations of both the above examples.

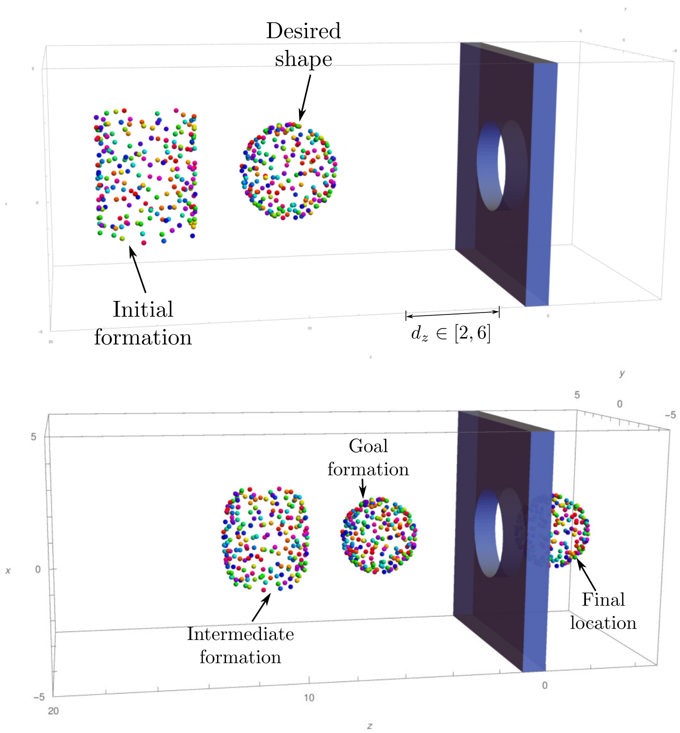

Fig. 6. A variable scale and translation example with 200 robots where the robot formation changes to pass through the circular opening in the wall. (Top) Initial cylindrical formation and desired spherical shape of the robot formation. (Bottom) An intermediate formation, the goal formation, and final location of the formation after passing through the circular opening.

VII. Conclusion

This paper presents algorithms to compute the optimal assignments and formation parameters for a team of robots from a given initial formation to a variable goal formation; here variable formation means that the desired shape of the formation is given, and its scale and location parameters must be optimized. We used the sum of squared robot travel distances as the objective function to be minimized. For the case of n identical spherical robots separated by $2\sqrt{2}R$ at their initial and goal positions, this objective ensures that the trajectories are collision free. We showed that the assignment with variable goal formation problem can be transformed to a linear sum assignment problem, which can be solved using the Hungarian algorithm. Thus using the presented approach, the assignment problem with variable scale and translation goal formations has the same $\mathcal{O}(n^3)$ time complexity as the assignment problem with fixed goal formations. Results from simulations on 200 and 600 robots show the algorithm is sufficiently fast for practical applications.

Our algorithm assumes that the environment is free of obstacles. One future direction is to compute the valid ranges of formation parameters in an environment with obstacles and optimize over them. Extension of the current kinematic robot model to dynamics models will also be explored. Future work also includes characterizing the variable goal formation when allowing rotation of the desired shape. Experiments on a team of robots are also planned.

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